KKKudm, Jan 2020

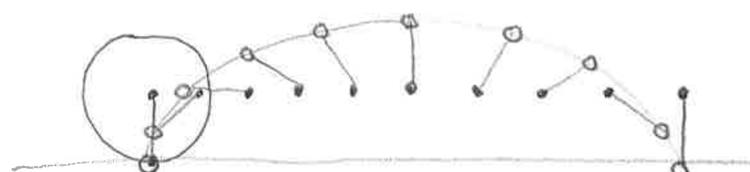
Aristotle's wheel (and the equations of a cycloid)

- We will not here be addressing the paradoxical questions regarding whether lines (and space itself) are continuous and infinitely divisible.

- These questions occupied some of the greatest minds in philosophy, physics and mathematics throughout history (Aristotle, Hero of Alexandria, Mersenne, Galileo, Torricelli, Boyle, Pascal, Dedekind, Cantor) ✯

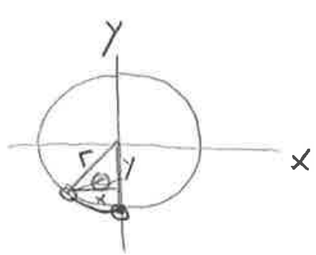
- We will simply use Aristotle's wheel as an opportunity to learn how to describe the trajectory of a point on a rolling wheel mathematically.

- Overview:



What is the mathematical formula for the trajectory of the dot on the wheel's perimeter?

- Start with a rotating (but not rolling) wheel



$$x = -r\sin\theta$$

$$y = -r\cos\theta$$

✯ See *Aristotle's Wheel: Notes on the History of a Paradox* by I.E. Drabkin (1939)

- If $\theta = \theta(t)$ (take, e.g. $\theta = \omega t$, where ω = angular velocity in $\frac{\text{radians}}{\text{second}}$ so θ is in radians)

then

$$x(t) = -r\sin(\omega t)$$

$$y(t) = -r(\cos(\omega t) - 1)$$

- As an aside, note that $s = r\theta$

so $\frac{ds}{dt} = r\frac{d\theta}{dt}$

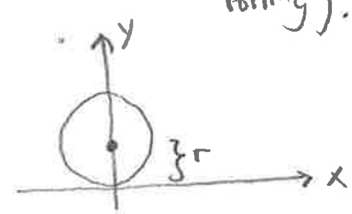
or $v_{dot} = \omega r$

- Now if the wheel is <u>rolling</u>, then we need to add the motion of the center of the wheel to the rotation about the center. The speed of the center of the wheel is, say, v_{cntr}

$$x_{center} = v_{cntr} t$$

$$x_{dot} = x_{center} - r\sin(\omega t)$$

move coord system down (so rolling).



So

$$x_{dot} = vt - r\sin(\omega t)$$

$$y_{dot} = r - r\cos(\omega t)$$

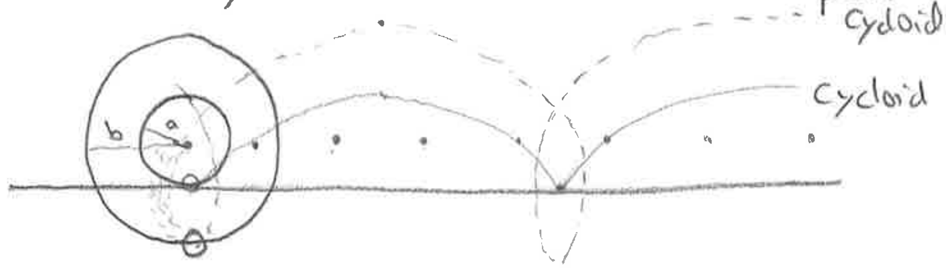
But $v_{center} = \omega r$, since the center moves at the same speed as the point touching the ground,

So

$$\boxed{\begin{aligned} x_{dot} &= r(\omega t - \sin(\omega t)) \\ y_{dot} &= r(1 - \cos(\omega t)) \end{aligned}}$$

Eqns of a cycloid!

- Challenge <u>exercise</u>: Describe the motion of a dot on a wheel hanging over a ledge whose axle is rolling. Axle radius a, wheel radius b.



- <u>Answer</u>

$$x = a\omega t - b\sin(\omega t)$$

$$y = a - b\cos(\omega t)$$