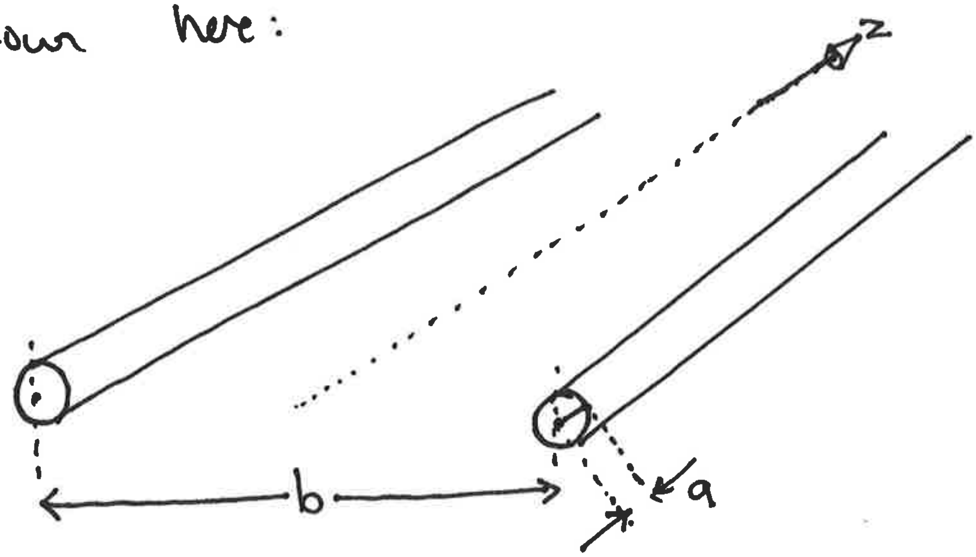
BB 5.3 (Parallel wire transmission line)

We wish to find the speed with which a signal propagates down a transmission line consisting of two long, parallel wires separated by a distance b, as shown here:



The wires have radius a. The signal can be represented by voltage and current waves:

$$V(z,t) = V_0 e^{i(\omega t - kz)}$$
$$I(z,t) = I_0 e^{i(\omega t - kz)}$$

where ω/k is the phase velocity, which can be written in terms of the capacitance per unit length, C_0, and the inductance per unit length, L_0.

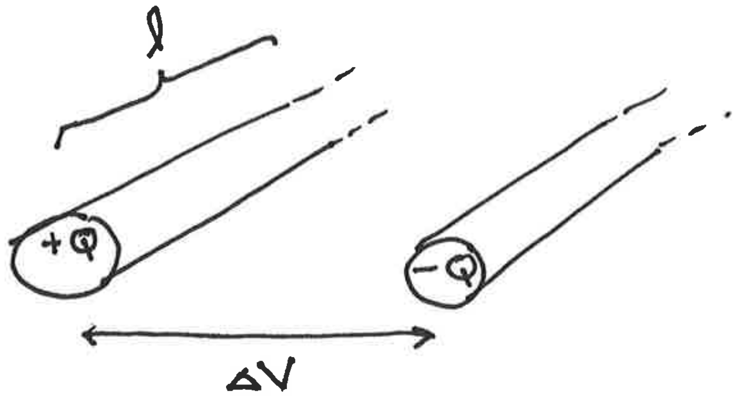
$$v_{phase} = \frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}}$$

- In order to compute this, we need to compute L_0 & C_0.

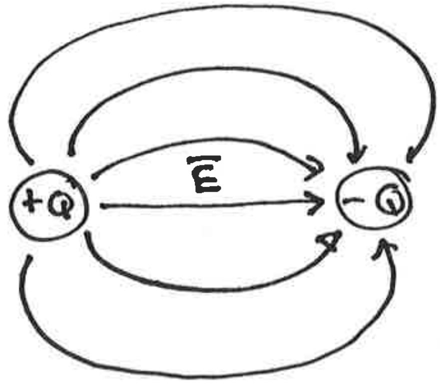
How? Recall that the self-inductance of a circuit element is the relationship between the current, I, flowing in the circuit and the magnetic flux through an (appropriately defined) region of the circuit, Φ. So $\boxed{L \equiv \frac{\Phi}{I}}$.

Also, recall that the capacitance of a circuit element is the relationship between the charge held by the circuit element, Q, and the voltage between two (appropriately defined) locations in the circuit element, V. So $\boxed{C = Q/V}$

- So starting with capacitance; let's find the voltage difference between the two wires when a charge density $\lambda = \frac{Q}{l}$ is placed on the wires. More specifically, when a charge $+Q$ is on the left wire and charge $-Q$ is placed on the right wire....



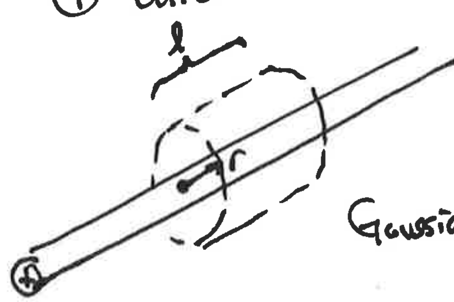
To find the voltage difference ΔV, we must integrate the electric field produced by the charges along a line between the wires. The electric fields look like this:



$\Delta V = -\int \bar{E} \cdot d\bar{s}$

line integral along path from $+Q$ to $-Q$.

What, though, is the electric field? To find this, we just use Gauss's law to find the field produced by each wire separately and then sum the electric fields. Consider the ⊕ wire



Gaussian surface at distance r_+

$$\oint_{\text{surf}} \bar{E} \cdot d\bar{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$E(r)\, 2\pi r_+ \ell = \frac{Q_{enc}}{\epsilon_0}$$

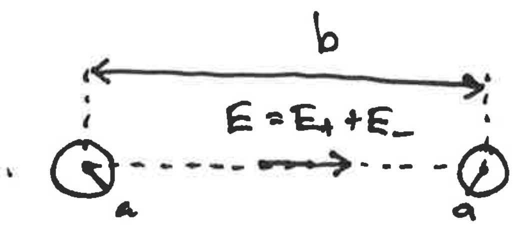
$$E(r) = \frac{Q_{enc}/\ell}{2\pi\epsilon_0 r_+}$$

- So the electric field at a distance r_+ from the positively charged wire is $E_+(r_+) = \dfrac{\lambda_+}{2\pi\epsilon_0 r_+}$ where λ_+ is the charge per length of the wire and E_+ is the electric field caused by the ⊕ charge.

- Similarly, at a distance r_- from the ⊖ charged wire

$$E_-(r_-) = \frac{\lambda_-}{2\pi\epsilon_0 r_-}$$

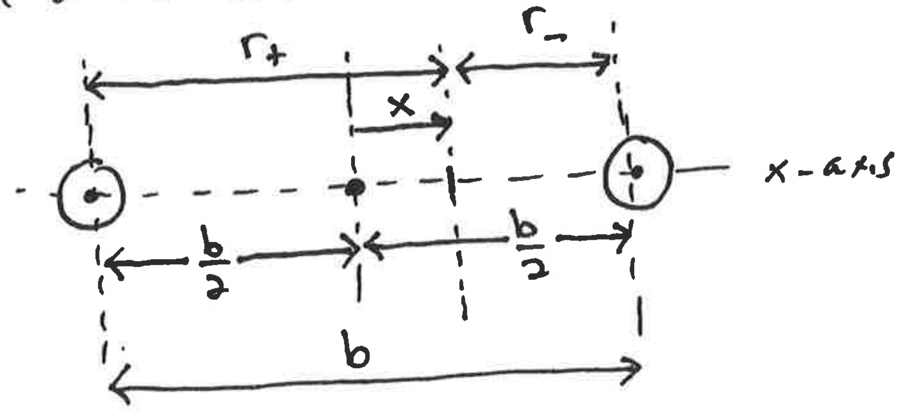
- We can now sum E_+ and E_- to find the total electric field E in the region around the wires. Then we can integrate the electric field along a path connecting the wires to find the voltage difference between the wires.



The voltage difference is $V = -\int_{path} \bar{E} \cdot d\bar{s}$

- We will choose a path directly connecting the two along the shortest route. The voltage calculated will be $V = -\int_{path} (E_+ + E_-) \cdot dx$

- We have to be a bit careful since the electric fields we computed are written in terms of the distances from each wire individually, but the path is along the x-axis, from the surface of one wire to the surface of the other. Consider:



$$r_+ = x + b/2 \qquad\qquad r_- = \frac{b}{2} - x$$

$$V = -\int_{x=-\frac{b}{2}+a}^{x=\frac{b}{2}-a} \left\{ E_+(r_+) + E_-(r_-) \right\} dx$$

$$V = -\int_{x=-\frac{b}{2}+a}^{\frac{b}{2}-a} \frac{\lambda}{2\pi\epsilon_0} \left\{ \frac{1}{\left(x+\frac{b}{2}\right)} + \frac{1}{\left(\frac{b}{2}-x\right)} \right\} dx$$

- This is an annoying integral. Is there an easier way?

Hmm.... yes!

- If the wires are equally charged, then I know that $V=0$ at the midpoint between the wires. Well, this doesn't really make it much easier, but I'll just integrate this using wolfram alpha....

charge per unit length

$$V = \frac{\lambda}{2\pi\epsilon_0} \cdot 4\tanh^{-1}\left(1-\frac{2a}{b}\right)$$

this is an inverse hyperbolic tangent

- Now, the capacitance is $C = \frac{Q}{V}$

So the capacitance per unit length is

$$\boxed{C_0 = \frac{2\pi\epsilon_0}{4\tanh^{-1}\left(1-\frac{2a}{b}\right)}}$$

- If $a \ll b$, we can approximate the inverse hyperbolic tangent using

$$\tanh^{-1}(1-\epsilon) = -\frac{1}{2}\log\left(\frac{\epsilon}{2}\right) - \frac{\epsilon}{4} - \frac{\epsilon^2}{16} - \ldots$$

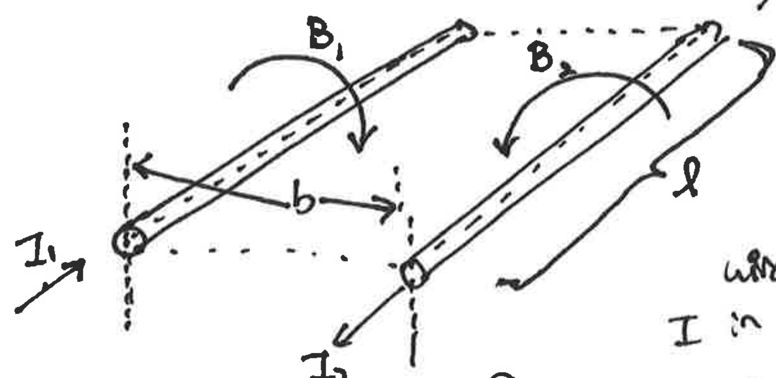
$$\tanh^{-1}\left(1-\frac{2a}{b}\right) \approx -\frac{1}{2}\log\left(\frac{a}{b}\right) = \frac{1}{2}\log\left(\frac{b}{a}\right)$$

- So our approximate capacitance per unit length for thin, widely seperated wires is

$$C_0 \approx \frac{\cancel{2}\pi\epsilon_0}{\cancel{4}\,\cancel{\tfrac{1}{2}}\log(b/a)} = \frac{\pi\epsilon_0}{\log(b/a)}$$

$$\boxed{C_0 \approx \frac{\pi\epsilon_0}{\log(b/a)}}$$ nice.

- Now, I need to find the inductance per unit length. I need to calculate the magnetic flux in the region between the wires that is caused by current I!



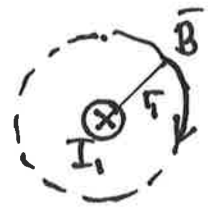
I'll compute the magnetic flux through the area between the wires caused by current I in each wire, as shown.

- What is the magnetic field caused by each wire? Then I'll sum them and integrate the field through the area.

$$\Phi = \int_{\text{area of loop.}} (\bar{B}_1 + \bar{B}_2)\cdot d\bar{A}$$

- To find the magnetic field caused by current I_1, use the ampere law $\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed}$

loop around wire



$$\int_0^{2\pi} B(r_1)\,\hat{\theta} \cdot r_1 d\theta\,\hat{\theta} = \mu_0 I_1$$

$$B(r_1)\, r_1 \int_0^{2\pi} d\theta = \mu_0 I_1$$

$$B(r_1) = \frac{\mu_0 I_1}{2\pi r}$$

- I'll just compute the flux of <u>this</u> magnetic field in the region between the wires.

$$\Phi_1 = \int \bar{B}(r_1) \cdot d\bar{A}$$

$$= \int_{r=a}^{r=b-a/2} \frac{\mu_0 I_1}{2\pi r}\, l\, dr$$

$$\Phi_1 = \frac{\mu_0 I_1 l}{2\pi} \ln\left(\frac{b - a/2}{a}\right)$$

- By symmetry, the other wire contributes the same flux. So the total flux is ....

$$\Phi = \Phi_1 + \Phi_2 = \frac{\cancel{2}\mu_0 I l}{\cancel{2}\pi} \log\left(\frac{b}{a} - \frac{1}{2}\right)$$

$$\Phi = \frac{\mu_0 I l}{\pi} \log\left(\frac{b}{a} - \frac{1}{2}\right)$$

- The inductance is $L = \frac{\Phi}{I}$ and the inductance per unit length is $L_0 = \frac{\Phi}{lI} = \frac{\mu_0}{\pi} \log\left(\frac{b}{a} - \frac{1}{2}\right)$

- Now, since $b \gg a$, $\log\left(\frac{b}{a} - \frac{1}{2}\right) \approx \log(b/a)$, so

$$\boxed{L_0 \approx \frac{\mu_0}{\pi} \log\left(\frac{b}{a}\right)}$$

- Now we can compute the velocity of a signal traveling down the transmission line.

$$v = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{\sqrt{\left[\frac{\mu_0}{\cancel{\pi}} \cancel{\log\left(\frac{b}{a}\right)}\right]\left[\frac{\cancel{\pi}\epsilon_0}{\cancel{\log(b/a)}}\right]}}$$

$$\boxed{v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

which is just the speed of light.

- Now, I'll compute the impedance.

$$Z_0 \equiv \sqrt{\frac{L_0}{C_0}}$$

$$= \sqrt{\frac{\frac{\mu_0}{\pi}\log\left(\frac{b}{a}\right)}{\frac{\pi \epsilon_0}{\log\left(\frac{b}{a}\right)}}}$$

$$= \sqrt{\frac{\mu_0 \log^2\left(\frac{b}{a}\right)}{\epsilon_0 \pi^2}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{\log\left(\frac{b}{a}\right)}{\pi}$$

$$\boxed{Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\,\log\left(\frac{b}{a}\right)}$$

- Finally, for #12 gauge wires, $2a = 0.0808''$

$b = 0.50''$

$\log\left(\frac{b}{a}\right) = 2.516$

$L_0 = 1.006\text{e-}6$

$C_0 = 1.106\text{e-}11$

$$\boxed{Z_0 = 302\ \Omega}$$